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A generalization of the Dijkgraaf-Witten invariant

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1 Introduction

In 1990 Dijkgraaf and Witten [6] introduced a topological invariant of closed oriented 3-manifolds using a finite group and its 3-cocycle. Let M be a closed oriented 3-manifold, G a finite group and $\alpha\in Z^{3}(BG, U(1))$. Then the Dijkgraaf-Witten invariant $Z(M)$ (we abbreviate it to the DW invariant in this paper) is defined as follows:

$$Z(M)=\frac{1}{|G|}\sum_{\gamma\in \mathrm{Hom}(\pi_{1}(M),G)}\langle\gamma^{*}[\alpha], [M]\rangle\in \mathbb{C}.$$

The topological invariance of $Z(M)$ is obvious from the definition and it is also evident that $Z(M)$ is a homotopy invariant since M only appears at the fundamental group and the fundamental class in the definition of $Z(M)$.

Dijkgraaf and Witten reformulated the invariant by using a triangulation of M in the following way. Let K be a triangulation of M. Then the fundamental class of M is described by the sum of the tetrahedra of K and $\gamma\in \mathrm{Hom}(\pi_{1}(M), G)$ is represented by assigning an element of G to each edge of K. $Z(M)$ is described as follows:

$$Z(M)=\frac{1}{|G|^{a}}\sum_{\varphi\in \mathrm{Col}(K)}\prod_{\text{tetrahedron}}\alpha(g, h, k)^{\pm 1},$$

where a is the number of the vertices of K and $g, h, k\in G$ are colors of edges of a tetrahedron of K. Wakui [12] proved the topological invariance of the DW invariant in this combinatorial construction. Due to the above construction of $Z(M)$ by using a triangulation, we can view the DW invariant as the "Turaev-Viro type"invariant.

This construction by using a triangulation enable us to define the DW invariant for a compact oriented 3-manifold M with $\partial M\neq\emptyset$. However, for $\partial M\neq\emptyset$ case, the DW invariant of M is determined not only by M but also by a triangulation of ∂M and its coloring.

Here we construct another version of the DW invariant, which we call the generalized DW invariant. For a compact oriented 3-manifold M with $\partial M \neq \emptyset$, the generalized DW invariant of M does not need a triangulation of ∂M nor its coloring. We can achieve that by using an ideal triangulation of a compact oriented 3-manifold with non-empty boundary or a cusped oriented 3-manifold. This is an analogy of the construction of the Turaev-Viro invariant in [2] for a compact 3-manifold with non-empty boundary or a cusped 3-manifold.

We calculate the generalized DW invariants for some examples and show that the invariants actually distinguish some pairs of cusped hyperbolic 3-manifolds with the same hyperbolic volumes and with the same Turaev-Viro invariants. We also give an example of a pair of cusped hyperbolic 3-manifolds with the same hyperbolic volumes and with the same homology groups, meanwhile with distinct generalized DW invariants.

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2 Definition of the generalized Dijkgraaf-Witten invariant

First we review the group cohomology briefly. Let G be a finite group and A a multiplicative abelian group. The n-cochain group $C^n(G, A)$ is defined as follows:

$$C^n(G,A) = \begin{cases} A & (n=0) \\ \{\alpha : \overbrace{G\times\cdots\times G}^{n} \to A\} & (n \geq 1). \end{cases}$$

The group operation of $C^n(G, A)$ is a multiplication of maps induced by the multiplication of A and then $C^n(G, A)$ is a multiplicative abelian group since so is A. The n-coboundary map $\delta^n : C^n(G, A) \to C^{n+1}(G, A)$ is defined by

$$(\delta^0 a)(g) = 1 \quad (a \in A,\ g \in G),$$

$$(\delta^n \alpha)(g_1, \ldots, g_{n+1}) =$$

$$\alpha(g_2, \ldots, g_{n+1})(\prod_{i=1}^{n} \alpha(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1})^{(-1)^i})\alpha(g_1, \ldots, g_n)^{(-1)^{n+1}},$$

$$(\alpha \in C^n(G, A),\ g_1, \ldots, g_{n+1} \in G,\ n \geq 1).$$

Then we can confirm by the above definition that $\{(C^n(G,A),\delta^n)\}_{n=0}^{\infty}$ is a cochain complex. Hence the n-cocycle group $Z^n(G,A)$ and the n-th cohomology group $H^n(G,A)$ are defined as usual.

An n-cochain $\alpha \in C^n(G,A)$ is said to be *normalized* if for any $g_1,\ldots,g_n \in G$, α satisfies

$$\alpha(1,g_2,\ldots,g_n)=\alpha(g_1,1,g_3,\ldots,g_n)=\cdots=\alpha(g_1,\ldots,g_{n-1},1)=1\in A.$$

If α and β are normalized n-cochains, $\alpha\beta$ and α^{-1} are also normalized n-cochains and $\delta^n\alpha$ is a normalized $(n+1)$-coboundary. Eilenberg and MacLane proved the following proposition [7, Lemma 6.1 and Lemma 6.2].

Proposition 2.1. *For any cochain α, there exists a normalized cochain α' which is cohomologous to α. For any normalized n-coboundary α, there exists a normalized (n-1)-cochain β such that $\alpha=\delta^{n-1}\beta$.*

Hence we assume that any n-cochain is normalized. As we only consider 3-cocycles in the rest of this paper, we restate the cocycle condition for a 3-cocycle α.

$$\alpha(h,k,l)\alpha(g,hk,l)\alpha(g,h,k)=\alpha(gh,k,l)\alpha(g,h,kl)\quad (g,h,k,l\in G).$$

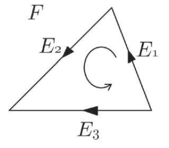
The cocycle condition takes an important role in the proof of the invariance of the genearalized DW invariant.

We can define the DW invariant by using any multiplicative abelian group A, nevertheless we usually use $U(1)$ in the definition of the original DW invariant. Hence we only consider $U(1)$-valued 3-cocycles in the rest of this paper.

In this paper we suppose that a triangulation K of a 3-manifold is not necessarily a decompositon as a simplicial complex. (A triangulation in this paper means a singular triangulation in [10] and [11].) For given four vertices of K, K may have more than one tetrahedron with the given four vertices. For given two vertices of K, there may exist more than one edge connecting the given two vertices. If a decomposition forms a simplicial complex, we call the decomposition *a simplicial triangulation*.

Let M be a compact oriented 3-manifold with boundary. We consider a triangulation of M with ideal vertices such that each boundary component of M converges at an ideal vertex. We call such a triangulation of M with ideal vertices *a generalized ideal triangulation* of M in this paper. In general, a generalized ideal triangulation K of M has both interior vertices and ideal vertices. If $\partial M=\emptyset$, K has no ideal vertices, that is, K is an ordinary triangulation of a closed 3-manifold M. On the other hand, an ideal triangulation is a generalized ideal triangulation without interior vertices.

Now we explain a coloring and a local order of a triangulation.



$\epsilon_1 = 1, \epsilon_2 = 1, \epsilon_3 = -1.$

Figure 1: The sign of edges.

Fix a generalized ideal triangulation K of M. Give an orientation to each edge and each face of K. *A coloring* φ of K is a map

$$\varphi : \{\text{oriented edges of } K\} \to G$$

satisfying

$$\varphi(E_3)^{\epsilon_3}\varphi(E_2)^{\epsilon_2}\varphi(E_1)^{\epsilon_1} = 1 \in G$$

for oriented edges E_1, E_2 and E_3 of any oriented 2-face F and

$$\epsilon_i = \begin{cases} 1 & \text{the orientation of } E_i \text{ agrees with that of } \partial F \\ -1 & \text{otherwise.} \end{cases}$$

(Note that the three edges E_1, E_2 and E_3 of F are chosen along the orientation of F as Figure 1.) The above condition for a coloring φ is required because a coloring φ originally comes from $\gamma \in \mathrm{Hom}(\pi_1(M), G)$. Let $\mathrm{Col}(K)$ be the set of the colorings of K. Note that a coloring φ of K is independent of the choice of orientations of edges and faces of K.

Fix a generalized ideal triangulation K of M. Give an orientation to each edge of K such that for any 2-face F of K, the orientations of the three edges of F are not cyclic (as the left hand side of Figure 2). We call such a choice of the orientations of edges of K *a local order of* K (or *a branching of* K). Then each tetrahedron σ of K has one of each vertex incident to i outgoing edges of σ and to $(3-i)$ incoming edges of σ for $i = 0, 1, 2, 3$ (as the right hand side of Figure 2). Let v_i be the vertex of σ incident to i outgoing edges of σ. Then the order $v_0 < v_1 < v_2 < v_3$ of the vertices of σ settles an orientation of σ. We define the sign ϵ_σ of σ as follows:

$$\epsilon_\sigma = \begin{cases} 1 & \text{the orientation of } \sigma \text{ by the local order agrees with that of } M \\ -1 & \text{otherwise.} \end{cases}$$

Now we define the generalized DW invariant. Let M be a compact or cusped 3-manifold, G a finite group and $\alpha \in Z^3(G, U(1))$. Fix a generalized ideal triangulation K of M with a local order. Then for each tetrahedron σ of K the sign ϵ_σ is determined by the

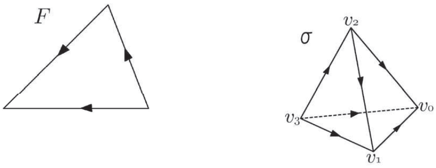


Figure 2: A local order for a face and for a tetrahedron.

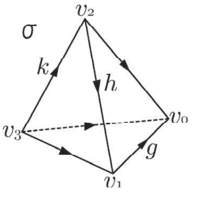


Figure 3: A colored tetrahedron.

local order. Put a coloring φ of K, and then some element $\varphi(E)$ of G is assigned to each oriented edge E of each tetrahedron σ. We call $\varphi(E)$ the color of E and such a tetrahedron σ the colored tetrahedron, denoted by (σ, φ). Let v_0, v_1, v_2, v_3 be the vertices of σ with $v_0 < v_1 < v_2 < v_3$ by the local order (v_i is incident to i outgoing edges of σ). Put $\varphi(\langle v_0 v_1 \rangle) = g$, $\varphi(\langle v_1 v_2 \rangle) = h$, $\varphi(\langle v_2 v_3 \rangle) = k$. Correspond $\alpha(g,h,k)^{\epsilon_\sigma} \in U(1)$ to the colored tetrahedron (σ, φ). We call $W(\sigma, \varphi) = \alpha(g,h,k)^{\epsilon_\sigma}$ the symbol of the colored tetrahedron (σ, φ).

Theorem 2.2. *Let M be a compact or cusped 3-manifold, G a finite group and $\alpha \in Z^3(G, U(1))$. Let K be a generalized ideal triangulation of M with a local order. Let $\sigma_1, \ldots, \sigma_n$ be the tetrahedra of K and a the number of the interior vertices of K. The generalized Dijkgraaf-Witten invariant $Z(M)$ is defined as follows:*

$$Z(M) = \frac{1}{|G|^a} \sum_{\varphi \in \mathrm{Col}(K)} \prod_{i=1}^{n} W(\sigma_i, \varphi).$$

Then $Z(M)$ is independent of the choice of a generalized ideal triangulation K of M with a local order.

By using a generalized ideal triangulation K of M, each component of ∂M corresponds to an ideal vertex of K. Hence, even if $\partial M \neq \emptyset$, the generalized DW invariant of M does not need a triangulation of ∂M nor its coloring. For a closed 3-manifold M, since K has no ideal vertices, the generalized DW invariant of M is no other than the original DW invariant of M.

Remark 2.3. In general some generalized ideal triangulation K of M does not admit a local order. Nevertheless the following lemma holds.

Lemma 2.4. *Any compact or cusped 3-manifold M has a generalized ideal triangulation which admits a local order.*

Proof. For any given generalized ideal triangulation K of M, let K^{bb} be the generalized ideal triangulation of M obtained by applying the barycentric subdivision twice to each tetrahedron of K. For given four vertices of K^{bb} (which form a tetrahedron of K^{bb}), there exists a unique tetrahedron of K^{bb} with the given four vertices. Hence K^{bb} can be dealt in the same way as a simplicial triangulation of a closed 3-manifold. We choose an arbitrary total order on the set of the vertices of K^{bb} and then the total order determines a local order of K^{bb}. □

3 Invariance of the generalized Dijkgraaf-Witten invariant

In this section, we prove Theorem 2.2. First we show that $Z(M)$ is independent of the choice of a local order of a fixed generalized ideal triangulation K of M. Then we prove that $Z(M)$ is independent of the choice of a generalized ideal triangulation K of M.

Let K be a generalized ideal triangulation of M with a local order. $\check{K}$ denotes the generalized ideal triangulation without considering a local order in this section. We define $Z(K)$ by

$$Z(K) = \frac{1}{|G|^a} \sum_{\varphi \in \mathrm{Col}(K)} \prod_{i=1}^{n} W(\sigma_i, \varphi).$$

Lemma 3.1. *Let K_1 and K_2 be generalized ideal triangulations with local orders of a compact or cusped 3-manifold M. If $\check{K}_1 = \check{K}_2$, then $Z(K_1) = Z(K_2)$, i.e. $Z(K)$ is independent of the choice of a local order.*

Proof. Let K be a generalized ideal triangulation of M with a local order. Let K^b be the generalized ideal triangulation of M obtained by applying the barycentric subdivision once to each tetrahedron of K with the following local order:

(vertex of K) $<$ (midpoint of an edge of K) $<$ (center of a face of K)
$<$ (center of a tetrahedron of K).

We prove that $Z(K) = Z(K^b)$, which implies the independence of the choice of a local order. We prove this claim by the following three steps.

Step 1 : Divide each tetrahedron σ of K into four tetrahedra by adding four edges connecting the center of σ (denoted by b) and (four) vertices of σ. This division is the number of the tetrahedra of K times of (1,4)-Pachner moves. See Figure 4. K' denotes the generalized ideal triangulation of M obtained by Step 1 with the local order

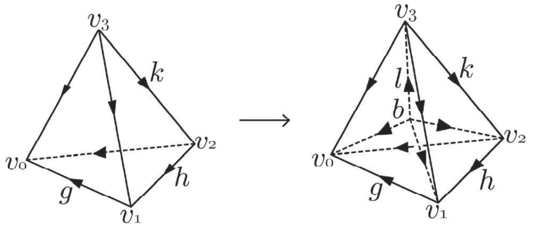


Figure 4: The division in Step 1 ((1,4)-Pachner move).

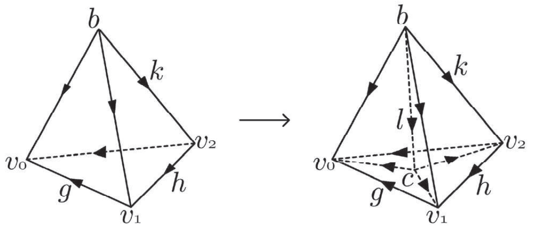


Figure 5: The division in Step 2.

(vertex of K) $<$ (center of a tetrahedron of K).

Step 2 : Divide each tetrahedron σ of K' into three tetrahedra by adding three edges as follows. σ has three vertices of K (the other vertex of σ is b). Let F be the face of σ with three vertices of K. Add three edges connecting the center of F (denoted by c) and (three) vertices of F. See Figure 5. K'' denotes the generalized ideal triangulation of M obtained by Step 2 with the local order
(vertex of K) $<$ (center of a face of K) $<$ (center of a tetrahedron of K).

Step 3 : Divide each tetrahedron σ of K'' into two tetrahedra as follows. Let v_0, v_1 be two vertices of σ which are vertices of K (the other vertices of σ are b and c). Let E be the edge of σ connecting v_0 and v_1, and d the midpoint of E. Divide $\sigma = \langle v_0 v_1 c b \rangle$ into $\langle v_0 d c b \rangle$ and $\langle v_1 d c b \rangle$. See Figure 6. The generalized ideal triangulation of M obtained by Step 3 is $\check{K}^b$.

Hence it suffices to show that $Z(K) = Z(K') = Z(K'') = Z(K^b)$. The proof of these equalities are given in [8].

□

Next we prove that $Z(M)$ is independent of the choice of a generalized ideal triangulation K of M. In order to show that, we make use of the following theorem by Pachner.

Theorem 3.2 (Pachner)**.** *Any two simplicial triangulations of a 3-manifold M can be transformed one to another by a finite sequence of the two types of transformations shown*

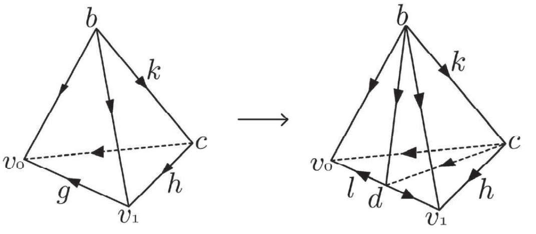


Figure 6: The division in Step 3.

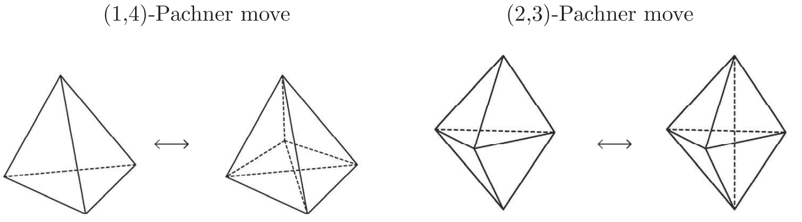


Figure 7: The Pachner moves.

in Figure 7.

Let K and L be any two generalized ideal triangulations of M. Owing to Lemma 3.1, $Z(K) = Z(L)$ implies Theorem 2.2.

Suppose K and L are simplicial. By Theorem 3.2, there exists a finite sequence of generalized ideal triangulations of M, $K = K_0 \to K_1 \to \cdots \to K_n = L$, such that K_i is transformed to K_{i+1} by one of Pachner moves once and each K_i is simplicial. Hence $Z(K_i) = Z(K_{i+1})$ for each i implies $Z(K) = Z(L)$. $Z(K_i) = Z(K_{i+1})$ follows from the following two lemmas given in [13].

Lemma 3.3. *If K_i is transformed to K_{i+1} by a* (1,4)*-Pachner move, then $Z(K_i) = Z(K_{i+1})$.*

Lemma 3.4. *If K_i is transformed to K_{i+1} by a* (2,3)*-Pachner move, then $Z(K_i) = Z(K_{i+1})$.*

Therefore if K and L are simplicial, $Z(K) = Z(L)$ holds.

Next we consider a generalized ideal triangulation K of M which is not simplicial. Let K^{bb} be the generalized ideal triangulation of M obtained by applying the barycentric subdivision to each tetrahedron of K twice. Even though K is not simplicial, K^{bb} is always simplicial. Furthermore, by Lemma 3.1, $Z(K) = Z(K^{bb})$ holds, which implies $Z(K) = Z(L)$ for any two generalized ideal triangulations of K and L of M.

This completes the proof of Theorem 2.2.

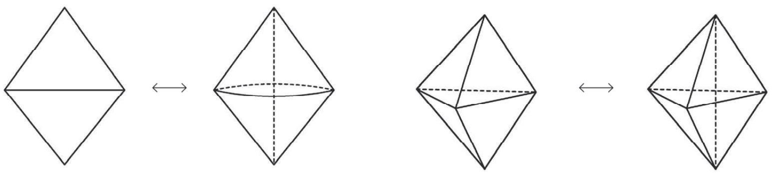


Figure 8: The Pachner moves for ideal triangulations.

We present simple properties of the generalized DW invariant which are known for the original DW invariant in [12]. The following proposition can be proved in the same way as the original DW case in [12].

Proposition 3.5. *Let M be a compact or cusped oriented 3-manifold, G a finite group and $\alpha \in Z^3(G, U(1))$. Then the following holds.*

(1) $Z(M)$ only depends on the cohomology class of α.

(2) $Z(-M) = \overline{Z(M)}$, where $-M$ is the oriented 3-manifold with the opposite orientation to M.

Although we introduce a generalized ideal triangulation in the definition of the generalized DW invariant, it suffices to consider ideal triangulations of M in calculations of $Z(M)$ by the following two theorems.

Theorem 3.6 ([10, Theorem 1.2.27])**.** *Any two ideal triangulations of a 3-manifold M can be transformed one to another by a finite sequence of the two types of transformations shown in Figure 8.*

We call a (2,3)-Pachner move that increases the number of the ideal tetrahedra *a positive (2,3)-Pachner move* in this paper. In general, a given ideal triangulation of M may not admit a local order. However Benedetti and Petronio proved the existence of an ideal triangulation with a local order [3, Theorem 3.4.9].

Theorem 3.7 (Benedetti-Petronio)**.** *Let M be a compact oriented 3-manifold with boundary and K an ideal triangulation of M. Then there exists a finite sequence of ideal triangulations of M, $K = K_0 \to K_1 \to \cdots \to K_n$, such that K_i is transformed to K_{i+1} by a positive (2,3)-Pachner move and K_n admits a local order.*

Corollary 3.8. *For any cusped or compact 3-manifold M with boundary, there exists an ideal triangulation K of M with a local order. Since K does not have interior vertices,*

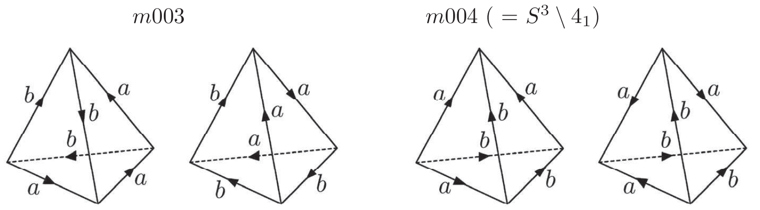


Figure 9: Minimal ideal triangulations of $m003$ and $m004$.

the generalized Dijkgraaf-Witten invariant $Z(M)$ is described by the following form:

$$Z(M) = \sum_{\varphi \in \mathrm{Col}(K)} \prod_{i=1}^{n} W(\sigma_i, \varphi).$$

4 Examples of cusped hyperbolic 3-manifolds

In this section, we calculate the generalized DW invariants of some cusped orientable hyperbolic 3-manifolds by using Theorem 3.7 and Corollary 3.8. We show that the generalized DW invariants distinguish some pairs of cusped hyperbolic 3-manifolds with the same hyperbolic volumes and with the same Turaev-Viro invariants. We also present an example of a pair of cusped hyperbolic 3-manifolds with the same hyperbolic volumes and with the same homology groups, whereas with distinct generalized DW invariants.

For a positive integer m, it is known that $H^3(\mathbb{Z}_m, U(1))$ is isomorphic to $\mathbb{Z}_m$ and a generator α of $H^3(\mathbb{Z}_m, U(1)) \cong \mathbb{Z}_m$ is described by the following formula [1]:

$$\alpha(g_1, g_2, g_3) = \exp\left(\frac{2\pi i}{m^2}\overline{g_1}(\overline{g_2} + \overline{g_3} - \overline{g_2 + g_3})\right),$$

where $\overline{g_i} \in \{0, \cdots, m-1\}$ is a representative of $g_i \in \mathbb{Z}_m$.

(1) $m003$ and $m004$

According to Regina [4] and SnapPy [5], $m003$ and $m004$ are cusped orientable 3-manifolds with the minimal ideal triangulations shown in Figure 9. The 3-manifold $m004$ is the figure eight knot complement. Their hyperbolic volumes, Turaev-Viro invariants and homology groups are as follows:

$$\mathrm{Vol}(m003) = \mathrm{Vol}(m004) \approx 2.02988,$$

$$TV(m003) = \sum_{(a,a,b),(a,b,b) \in adm} w_a w_b \begin{vmatrix} a & a & b \\ a & b & b \end{vmatrix} \begin{vmatrix} a & a & b \\ a & b & b \end{vmatrix} = TV(m004),$$

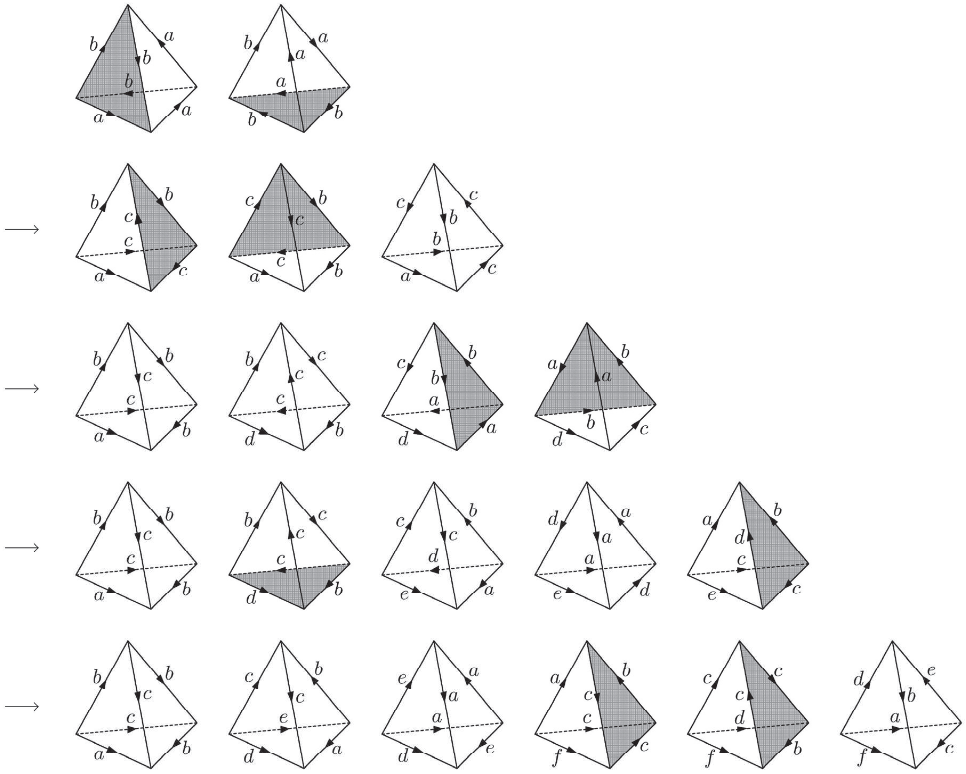
$$H_1(m003; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_5, \quad H_1(m004; \mathbb{Z}) = \mathbb{Z}.$$

We show that $m003$ and $m004$ have distinct generalized DW invariants.

First we calculate the generalized DW invariant of $m004$. The minimal ideal triangulation of $m004$ admits the local order shown in Figure 9. Identify the labels of edges with the colors of edges. By the left front face of the left ideal tetrahedron of $m004$ shown in Figure 10, $a = ba$. By the right front face of the left ideal tetrahedron of $m004$, $b = ab$. Hence $a = b = 1 \in G$, which implies $m004$ has only a trivial coloring. Therefore, for any finite group G and its any normalized 3-cocycle α,

$$Z(m004) = 1.$$

On the other hand, the minimal ideal triangulation of $m003$ shown in Figure 9 does not admit a local order. Then we apply Theorem 3.7 to the ideal triangulation of $m003$. In order to assign a local order, transform the ideal triangulation of $m003$ by positive (2,3)-Pachner moves.



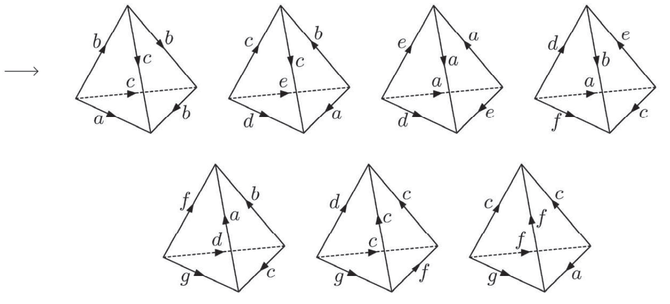


Figure 10: A sequence of (2,3)-Pachner moves for $m003$ to obtain a locally ordered ideal triangulation.

After positive (2,3)-Pachner moves five times, the ideal triangulation of $m003$ which consists of seven ideal tetrahedra admits the local order shown in Figure 10. The relations between the colors of edges are the following:

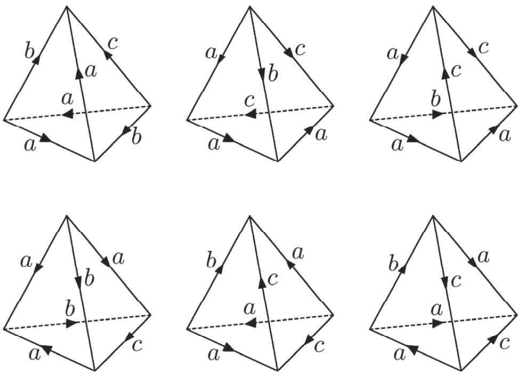
$$a = b^3,\ c = b^2,\ d = b^4,\ e = b,\ f = 1,\ g = b^2,\ b^5 = 1.$$

$$\begin{aligned} Z(m003) = \sum_{b \in G, b^5 = 1} & \alpha(b,b,b)^{-1}\alpha(b^2,b,b)\alpha(b^3,b^3,b^3) \\ & \times \alpha(b,b,b^3)\alpha(b,b^2,b^2)\alpha(b^2,b^3,b^2). \end{aligned}$$

In order to confirm $Z(m003) \neq Z(m004)$, we calculate $Z(m003)$ for $G = \mathbb{Z}_5$ and a generator α of $H^3(\mathbb{Z}_5, U(1)) \cong \mathbb{Z}_5$.

$$\begin{aligned} Z(m003) = 1 &+ \exp\left(\frac{2\pi i}{5}(3+2)\right) + \exp\left(\frac{2\pi i}{5}2(1+2)\right) + \exp\left(\frac{2\pi i}{5}3(-1+2+3+1+2)\right) \\ &+ \exp\left(\frac{2\pi i}{5}4(-1+2+1+1+2)\right) \\ = 3 &+ 2\exp\frac{2\pi i}{5} \\ = \frac{1}{2}&\left(5 + \sqrt{5} + i\sqrt{10 + 2\sqrt{5}}\right). \end{aligned}$$

Hence the generalized DW invariants distinguish $m003$ and $m004$.



$m006$

$m007$

Figure 11: Minimal ideal triangulations of $m006$ and $m007$.

(2) $m006$ and $m007$

According to Regina [4] and SnapPy [5], $m006$ and $m007$ are cusped orientable 3-manifolds with the minimal ideal triangulations shown in Figure 11. Their hyperbolic volumes, Turaev-Viro invariants and homology groups are as follows:

$$\mathrm{Vol}(m006) = \mathrm{Vol}(m007) \approx 2.56897,$$

$$TV(m006) = \sum w_a w_b w_c \begin{vmatrix} a & b & c \\ a & b & a \end{vmatrix} \begin{vmatrix} a & b & c \\ a & c & a \end{vmatrix} \begin{vmatrix} a & b & c \\ a & c & a \end{vmatrix} = TV(m007),$$

$$H_1(m006;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_5, \quad H_1(m007;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_3.$$

$$Z(m006) = \sum_{a \in G, a^5=1} \alpha(a,a,a)^3 \alpha(a,a^2,a)\alpha(a^3,a^3,a^3).$$

$$Z(m007) = \sum_{a \in G, a^3=1} \alpha(a,a,a)\alpha(a^{-1},a^{-1},a^{-1}).$$

If $G = \mathbb{Z}_5$ and α is a generator of $H^3(\mathbb{Z}_5, U(1)) \cong \mathbb{Z}_5$,

$$\begin{aligned} Z(m006) = 1 &+ \exp\left(\frac{2\pi i}{5} \times 3\right) + \exp\left(\frac{2\pi i}{5} \times 2 \times 1\right) + \exp\left(\frac{2\pi i}{5} 3(3+3)\right) \\ &+ \exp\left(\frac{2\pi i}{5} 4(3+1)\right) \\ &= 1 + 2\exp\frac{6\pi i}{5} + \exp\frac{4\pi i}{5} + \exp\frac{2\pi i}{5} \\ &= -\frac{\sqrt{5}}{2} + \frac{i}{4}\left(\sqrt{10+2\sqrt{5}} - \sqrt{10-2\sqrt{5}}\right), \end{aligned}$$

$$Z(m007) = 1.$$

Hence the generalized DW invariants distinguish $m006$ and $m007$.

In fact the previous two pairs of cusped hyperbolic 3-manifolds with the same hyperbolic volumes and the same Turaev-Viro invariants are distinguished by their homology groups. The following pair of cusped hyperbolic 3-manifolds with the same hyperbolic volumes and the same homology groups have distinct generalized DW invariants.

$s778$

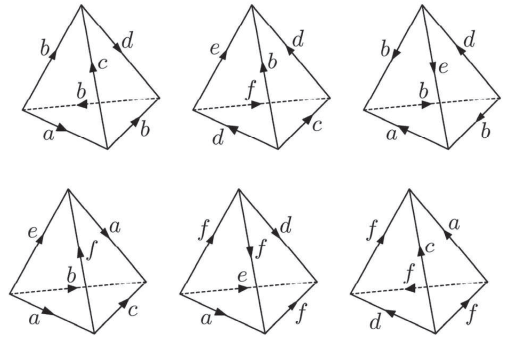


Figure 12: A minimal ideal triangulation of $s778$.

$s788$

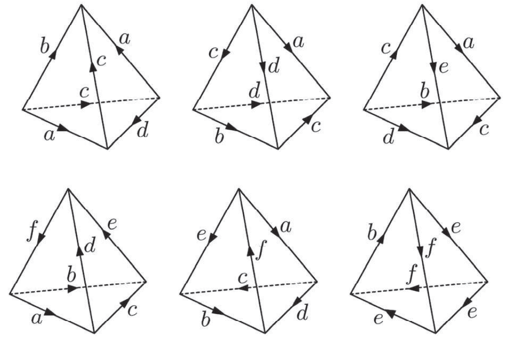


Figure 13: A minimal ideal triangulation of $s788$.

(3) $s778$ and $s788$

According to Regina [4] and SnapPy [5], $s778$ and $s788$ are cusped orientable 3-manifolds with the minimal ideal triangulations shown in Figure 12 and 13 respectively. Their hyperbolic volumes, homology groups and $SO(3)$ Turaev-Viro invariants [9] at $r = 5$ are as follows:

$$\mathrm{Vol}(s778) = \mathrm{Vol}(s788) \approx 5.33349,$$

$$H_1(s778;\mathbb{Z}) = H_1(s788;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_{12},$$
$$TV(s778) = 6 - 2\sqrt{5}, \quad TV(s788) = \frac{5-\sqrt{5}}{2}.$$

The minimal ideal triangulations of $s778$ and $s788$ shown in Figure 12 and 13 do not admit a local order. In order to assign a local order, transform the ideal triangulations of $s778$ and $s788$ by positive (2,3)-Pachner moves.

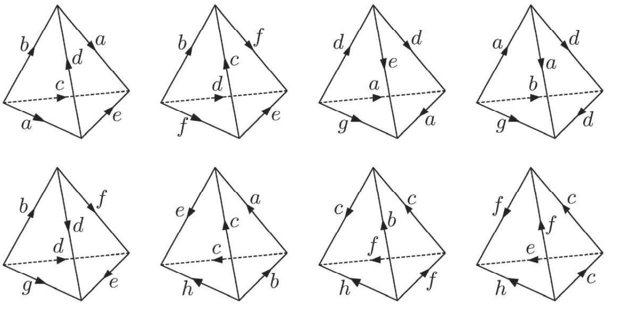


Figure 14: An ideal triangulation of $s778$ with a local order.

After positive (2,3)-Pachner moves twice, the ideal triangulation of $s778$ which consists of eight ideal tetrahedra admits the local order shown in Figure 14. The relations between the colors of edges are the following:

$$a = d^2,\ b = e = d^3,\ c = d^5,\ f = d^{10},\ g = d^4,\ h = d^8,\ d^{12} = 1.$$

$$\begin{aligned} Z(s778) = \sum_{d \in G, d^{12}=1} & \alpha(d,d,d^2)\alpha(d^2,d,d)\alpha(d^2,d,d^2)\alpha(d^3,d^2,d^3) \\ & \times \alpha(d^3,d^{10},d^3)\alpha(d^5,d^5,d^{10})\alpha(d^{10},d^5,d^5)\alpha(d^{10},d^5,d^{10}). \end{aligned}$$

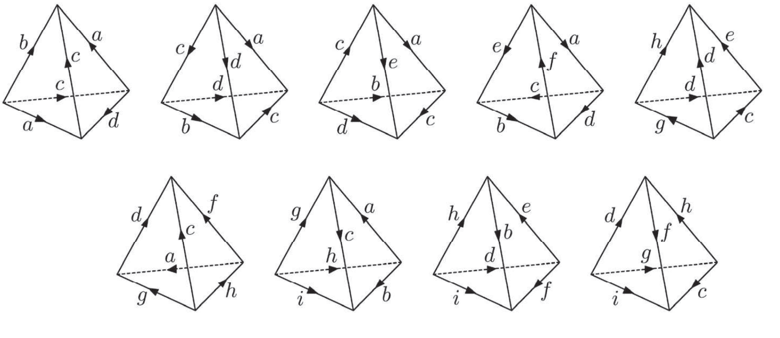


Figure 15: An ideal triangulation of $s788$ with a local order.

After positive (2,3)-Pachner moves three times, the ideal triangulation of $s788$ which consists of nine ideal tetrahedra admits the local order shown in Figure 15. The relations between the colors of edges are the following:

$$b = e = a^9,\ c = a^8,\ d = a^5,\ f = a^6,\ g = a^3,\ h = a^2,\ i = a^{-1},\ a^{12} = 1.$$

$$Z(s788) = \sum_{a \in G, a^{12}=1} \alpha(a^5, a, a^2)\alpha(a^6, a^2, a^3)\alpha(a^8, a, a^2)\alpha(a^8, a, a^8)^{-1}$$
$$\times\alpha(a^8, a^5, a^8)^{-1}\alpha(a^8, a^9, a^8)^{-1}\alpha(a^9, a^5, a^3)^{-1}\alpha(a^9, a^8, a)^{-1}\alpha(a^9, a^9, a^5).$$

In order to confirm $Z(s778) \neq Z(s788)$, we calculate $Z(s778)$ and $Z(s788)$ for $G = \mathbb{Z}_{12}$ and a generator α of $H^3(\mathbb{Z}_{12}, U(1)) \cong \mathbb{Z}_{12}$.

$$Z(s778) = -6, \quad Z(s788) = 3 - 2\sqrt{3}.$$

Hence the generalized DW invariants distinguish $s778$ and $s788$.

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